

SUPERSYMMETRY AND SPIN STRUCTURES

JOSÉ FIGUEROA-O'FARRILL AND SUNIL GADHIA

ABSTRACT. We construct examples of isometric M-theory backgrounds which preserve a different amount of supersymmetry depending on the choice of spin structure. These examples are of the form $\text{AdS}_4 \times L$, where L is a seven-dimensional lens space whose fundamental group is cyclic of order $4k$.

1. INTRODUCTION

The purpose of this brief note is to highlight and illustrate the importance of specifying the spin structure as part of the data defining a supergravity background. For every positive integer k we will construct pairs of M-theory backgrounds with the same geometry and flux but with two different spin structures and such that the amount of supersymmetry which is preserved depends on the choice of spin structure. We will construct these backgrounds by quotienting the Freund–Rubin background $\text{AdS}_4 \times S^7$ [1] of eleven-dimensional supergravity [2, 3] by a cyclic group of order $4k$ acting freely on the sphere.

The precise geometry of the Freund–Rubin background is

$$\text{AdS}_4(-8s) \times S^7(7s) \quad \text{with flux} \quad F = \sqrt{6s} \, \text{dvol}(\text{AdS}_4)$$

where the parameter $s > 0$ is the negative eleven-dimensional scalar curvature $R = -s$ and the numbers in parenthesis are the scalar curvatures of each of the geometries, this being the only modulus in a manifold of constant sectional curvature.

The supersymmetry of this background boils down to the existence of geometric Killing spinors in each factor:

$$\begin{aligned} \nabla_X \phi &= \frac{1}{6} f X \cdot \phi & \text{on } \text{AdS}_4 \\ \nabla_X \psi &= -\frac{1}{12} f X \cdot \psi & \text{on } S^7, \end{aligned}$$

where $f = \sqrt{6s}$. The space of Killing spinors is 32-dimensional and as a representation of the isometry Lie algebra $\mathfrak{so}(2, 3) \oplus \mathfrak{so}(8)$ of the background it is isomorphic to

$$\mathfrak{S}^{2,3} \otimes \mathfrak{S}_-^{8,0},$$

where $\mathfrak{S}^{p,q}$ denotes the half-spin representation of $\mathfrak{so}(p, q)$ and the subscript denotes chirality, if relevant. The chirality of the 8-dimensional half-spin representation has to do with the sign of the “Killing constant” f in the above Killing spinor equations. Changing the sign of f is tantamount to changing the sign of F which in turn is tantamount to reversing the orientation of the sphere. We will use this device below to simplify the discussion in certain points. The lesson is thus not so much that the 8-dimensional spinor representation has negative chirality, but that it *is* chiral.

A cyclic group $\Gamma \subset \text{SO}(8)$ acts isometrically on the round sphere $S^7 \subset \mathbb{R}^8$ by restricting to the sphere the linear action on \mathbb{R}^8 . We will assume that Γ acts freely on the sphere, so that no element

(except the identity) fixes a point in the sphere. In this case, the quotient S^7/Γ is smooth and locally isometric to S^7 : it is called a lens space. In particular it has constant positive sectional curvature, hence it is a spherical space form. The determination of all spherical space forms has a long history culminating in Wolf's solution [4].

As explained, for example, in [5, 6], the quotient S^7/Γ will admit a spin structure if and only if the action of Γ on the orthonormal frame bundle of S^7 lifts to the spin bundle. The total space of the orthonormal frame bundle of the round sphere S^7 is the Lie group $\mathrm{SO}(8)$, which fibres over S^7 with typical fibre $\mathrm{SO}(7)$. Indeed, S^7 can be thought of as the homogeneous space $\mathrm{SO}(8)/\mathrm{SO}(7)$. The action of $\Gamma \subset \mathrm{SO}(8)$ on the orthonormal frame bundle is simply left multiplication in the group $\mathrm{SO}(8)$ itself.

The total space of the spin bundle of S^7 is the spin group $\mathrm{Spin}(8)$ which fibres over S^7 with fibre $\mathrm{Spin}(7)$. We will let $\theta : \mathrm{Spin}(8) \rightarrow \mathrm{SO}(8)$ denote the two-to-one covering map. The action of Γ on $\mathrm{SO}(8)$ will lift to the spin bundle if and only if there exists a subgroup $\hat{\Gamma} \subset \mathrm{Spin}(8)$ which is mapped isomorphically to Γ under θ . The action of such $\hat{\Gamma}$ on the spin bundle is via left multiplication on the spin group itself. Thus we see that the spin structures in the quotient S^7/Γ are in one-to-one correspondence with the isomorphic lifts $\hat{\Gamma} \subset \mathrm{Spin}(8)$ of Γ .

A cyclic group Γ is specified by exhibiting a generator A . If Γ has order n , then $A^n = 1$. We will investigate the existence of the subgroup $\hat{\Gamma} \subset \mathrm{Spin}(8)$ by lifting the generator A to $\hat{A} \in \mathrm{Spin}(8)$, there being two such lifts distinguished by a sign, and then checking whether there exists a choice of sign for which the relation $\hat{A}^n = 1$ is satisfied in $\mathrm{Spin}(8)$, thus recovering an isomorphic group. We will work with $\mathrm{Spin}(8)$ inside the Clifford algebra $\mathcal{C}\ell(8)$, where in our conventions the Clifford product obeys $\mathbf{v}^2 = -|\mathbf{v}|^2 \mathbf{1}$ for $\mathbf{v} \in \mathbb{R}^8$, or in terms of gamma matrices,

$$\gamma_i \gamma_j + \gamma_j \gamma_i = -2\delta_{ij} \mathbf{1} .$$

The map $\theta : \mathrm{Spin}(8) \rightarrow \mathrm{SO}(8)$ is given explicitly in terms of the Clifford algebra as follows. Recall that $\mathrm{Spin}(8)$ embeds in the Clifford algebra as the product of even number of elements of $S^7 \subset \mathbb{R}^8$:

$$\mathrm{Spin}(8) = \{ \mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_{2k} \mid \mathbf{v}_i \in \mathbb{R}^8, |\mathbf{v}_i| = 1 \} .$$

If $\mathbf{v} \in \mathbb{R}^8$ and $s = \mathbf{v}_1 \mathbf{v}_2 \cdots \mathbf{v}_{2k} \in \mathrm{Spin}(8)$, then

$$\theta(s) \cdot \mathbf{v} = s \mathbf{v} \hat{s} ,$$

where $\hat{s} = \mathbf{v}_{2k} \cdots \mathbf{v}_1$. Since for $\mathbf{w} \in S^7$, $\mathbf{v} \mapsto \mathbf{w} \mathbf{v} \mathbf{w} = \mathbf{v} - 2\langle \mathbf{v}, \mathbf{w} \rangle \mathbf{w}$ is the reflection in the hyperplane perpendicular to \mathbf{w} , we see that this formula exhibits the action of $\mathrm{SO}(8)$ on \mathbb{R}^8 as a composition of reflections. In particular it is clear from this observation that if $\mathbf{v} \in \mathbb{R}^8$ then so is $s \mathbf{v} \hat{s}$, as claimed.

If $\Gamma \subset \mathrm{SO}(8)$ lifts isomorphically to $\hat{\Gamma} \subset \mathrm{Spin}(8)$, then we can investigate whether S^7/Γ admits any Killing spinors. Bär's cone construction [7] relates Killing spinors on S^7 to parallel spinors on \mathbb{R}^8 , which are themselves in one-to-one correspondence with the relevant half-spin representation of $\mathrm{Spin}(8)$. Moreover this correspondence is equivariant with respect to the isometry group. Therefore Killing spinors on S^7/Γ are in one-to-one correspondence with $\hat{\Gamma}$ -invariant parallel spinors in \mathbb{R}^8 , or equivalently with $\hat{\Gamma}$ -invariant spinors in the relevant half-spin representation of $\mathrm{Spin}(8)$. What this means in practise is that we must check, for each isomorphic lift $\hat{\Gamma}$ —that is, for each inequivalent spin structure in the quotient—whether $\hat{\Gamma}$ preserves any spinors in the half-spin representations \mathfrak{S}_{\pm}^8 .

2. SEVEN-DIMENSIONAL LENS SPACES

Every cyclic subgroup $\Gamma \subset \text{SO}(8)$ is conjugate (perhaps by $\text{O}(8)$) to $\Gamma(n, a, b, c)$, a cyclic subgroup of order n generated by

$$A = \begin{pmatrix} R\left(\frac{1}{n}\right) & & & \\ & R\left(\frac{a}{n}\right) & & \\ & & R\left(\frac{b}{n}\right) & \\ & & & R\left(\frac{c}{n}\right) \end{pmatrix},$$

where $R(\theta)$ denotes the rotation matrix

$$R(\theta) = \begin{pmatrix} \cos 2\pi\theta & \sin 2\pi\theta \\ -\sin 2\pi\theta & \cos 2\pi\theta \end{pmatrix},$$

and where $(a, n) = (b, n) = (c, n) = 1$. Without loss of generality we can order them so that $1 \leq a \leq b \leq c < n$. In choosing a, b, c positive we may have used conjugation by $\text{O}(8)$. Such conjugations may change the orientation of the sphere, which in turn change the Killing constant in the Killing spinor equation on S^7 or, equivalently, the chirality of the parallel spinors in \mathbb{R}^8 . What this means in practise is that we must consider *both* half-spin representations \mathfrak{S}_{\pm}^8 .

There are two possible lifts of A to $\text{Spin}(8) \subset \mathcal{C}\ell(8)$, distinguished by a sign ε :

$$\hat{A} = \varepsilon \exp \left(\frac{\pi}{n} \gamma_{12} + \frac{a\pi}{n} \gamma_{34} + \frac{b\pi}{n} \gamma_{56} + \frac{c\pi}{n} \gamma_{78} \right),$$

obeying

$$\hat{A}^n = \varepsilon^n (-1)^{1+a+b+c} \mathbb{1}.$$

We distinguish two cases. If n is even, then a, b, c are odd and hence $1 + a + b + c$ is even. Therefore $\hat{A}^n = \mathbb{1}$ for either choice of ε . Therefore there are two inequivalent spin structures in the corresponding quotient. If n is odd, we choose $\varepsilon = (-1)^{1+a+b+c}$, whence there is a unique spin structure in the quotient.

The eigenvalues of \hat{A} in the Clifford module are given by

$$\varepsilon \exp \left(\frac{i\pi}{n} (\sigma_1 + a\sigma_2 + b\sigma_3 + c\sigma_4) \right)$$

with σ_i signs whose product $\sigma_1\sigma_2\sigma_3\sigma_4$ determines the chirality. Since $1 \leq a \leq b \leq c < n$, the only way that this can be equal to 1 is if

$$\sigma_1 + a\sigma_2 + b\sigma_3 + c\sigma_4 = 0, \pm n, \pm 2n$$

depending on the value of ε : $0, \pm 2n$ for $\varepsilon = 1$ and $\pm n$ for $\varepsilon = -1$. In practise we do not have to worry about this dichotomy, because the existence of an invariant spinor implies the existence of a spin structure in the quotient, as explained, for example, in [5, Section 5.2].

Let us consider some cases as a way of illustration.

2.1. $n = 2$. Here we have only one possible choice $a = b = c = 1$, and the resulting geometry is $\text{AdS}_4 \times \mathbb{RP}^7$. We will have an invariant spinor whenever the weights σ_i add up to $0, \pm 2, \pm 4$. For the “positive” spin structure, we require the sum to be either 0 or ± 4 . This happens for the following choices of weights: $\pm \pm \pm \pm, \pm \pm \mp \mp, \pm \mp \pm \mp$ and $\pm \mp \mp \pm$, for a total of 8 all of positive chirality. For the “negative” spinor structure we require the sum to equal ± 2 . This happens for $\pm \pm \pm \mp, \pm \pm \mp \pm, \pm \mp \pm \pm$ and $\pm \mp \mp \mp$ for a total of 8 all of negative chirality. We conclude that this quotient preserves *all* of the supersymmetry of the vacuum.

It was proved by Franc [8] that of all the lens spaces, \mathbb{RP}^{4k+3} is the only one (apart from the sphere itself) admitting the maximal number of Killing spinors and it was later proved by Bär [9] that this is still the case among all spherical space forms.

2.2. $n = 3$. The possible choices for (a, b, c) are $(1, 1, 1)$, $(1, 1, 2)$ and $(1, 2, 2)$. The other choice $(2, 2, 2)$ gives the same quotient as $(1, 1, 2)$ since they generate conjugate subgroups. Let us take each case in turn.

2.2.1. $(1, 1, 1)$. We require $\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 0$, which happens for 6 weights $\pm\pm\mp\mp$, $\pm\mp\pm\mp$ and $\pm\mp\mp\pm$, all of positive chirality.

2.2.2. $(1, 1, 2)$. Here we require $\sigma_1 + \sigma_2 + \sigma_3 + 2\sigma_4 = \pm 3$, which happens for 6 weights $\mp\pm\pm\pm$, $\pm\mp\pm\pm$, and $\pm\pm\mp\pm$, all of negative chirality.

2.2.3. $(1, 2, 2)$. Here we can have $\sigma_1 + \sigma_2 + 2\sigma_3 + 2\sigma_4 = 0, \pm 6$, which happens for 6 weights $\pm\pm\pm\pm$, $\pm\mp\pm\mp$ and $\pm\mp\mp\pm$, all of positive chirality.

2.3. $n = 4$. Again there are three possible choices for (a, b, c) : $(1, 1, 1)$, $(1, 1, 3)$ and $(1, 3, 3)$, with $(3, 3, 3)$ and $(1, 1, 3)$ generating conjugate subgroups.

2.3.1. $(1, 1, 1)$. Here we can have $\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 0, \pm 4$, with those weights adding up to 0 and those to ± 4 in different spin structures. For the positive spin structure, they must add up to zero and there are six such weights of all positive chirality: $\pm\pm\mp\mp$, $\pm\mp\pm\mp$ and $\pm\mp\mp\pm$. For the negative spin structure, they must add up to ± 4 and there are two such weights $\pm\pm\pm\pm$ all of positive chirality.

2.3.2. $(1, 1, 3)$. Again we can have $\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 0, \pm 4$. For the positive spin structure, the sum must give 0 which happens for two negative-chirality weights: $\pm\pm\pm\mp$. For the negative spin structure, the sum must give ± 4 which happens for 6 negative-chirality weights: $\mp\pm\pm\pm$, $\pm\mp\pm\pm$ and $\pm\pm\mp\pm$.

2.3.3. $(1, 3, 3)$. Here we can have $\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 = 0, \pm 4, \pm 8$. For the positive spin structure the sum must either be 0 or ± 8 , which happens for 6 positive-chirality weights: $\pm\pm\pm\pm$, $\pm\mp\pm\mp$ and $\pm\mp\mp\pm$. For the negative spin structure, the sum must be ± 4 and this happens for two positive-chirality weights: $\pm\pm\mp\mp$.

Either one of these quotients constitutes possibly the simplest example of the phenomenon which we would like to illustrate: the same geometry $\text{AdS}_4 \times (S^7/\mathbb{Z}_4)$ preserves a different amount of supersymmetry depending on the choice of spin structure, in this case either $\frac{1}{4}$ or $\frac{3}{4}$.

2.4. $n = 4k \geq 8$. For $n = 4k$, $k > 1$, it is easily seen that the quotients S^7/\mathbb{Z}_{4k} with weights (a, b, c) given by $(1, 2k-1, 2k-1)$, $(1, 2k+1, 2k+1)$, $(2k-1, 2k+1, 4k-1)$ have four invariant spinors with respect to the positive spin structure and two with respect to the negative spin structure, the chiralities being the same in both cases. Similarly, the quotient with weight $(1, 2k-1, 2k+1)$ has two invariant spinors with respect to the positive spin structure and four with respect to the negative spin structure, again with chiralities agreeing. We conclude that the corresponding supergravity backgrounds $\text{AdS}_4 \times (S^7/\mathbb{Z}_{4k})$ are either $\frac{1}{2}$ -BPS or $\frac{1}{4}$ -BPS, depending on the spin structure.

2.4.1. $(1, 2k-1, 2k-1)$. In this case the spinors with weights $\pm\mp\pm\mp$ and $\pm\mp\mp\pm$ are invariant relative to the positive spin structure, whereas the spinors with weights $\pm\pm\pm\pm$ are invariant relative to the negative spin structure. All have positive chirality.

2.4.2. $(1, 2k+1, 2k+1)$. In this case the spinors with weights $\pm \mp \pm \mp$ and $\pm \mp \mp \pm$ are invariant relative to the positive spin structure, whereas the spinors with weights $\pm \pm \mp \mp$ are invariant relative to the negative spin structure. All have positive chirality.

2.4.3. $(2k-1, 2k+1, 4k-1)$. In this case the spinors with weights $\pm \pm \pm \pm$ and $\pm \mp \mp \pm$ are invariant relative to the positive spin structure, whereas the spinors with weights $\pm \pm \mp \mp$ are invariant relative to the negative spin structure. All have positive chirality.

2.4.4. $(1, 2k-1, 2k+1)$. In this case the spinors with weights $\pm \pm \pm \mp$ are invariant relative to the positive spin structure, whereas the spinors with weights $\pm \mp \pm \pm$ and $\mp \pm \pm \pm$ are invariant relative to the negative spin structure. All have negative chirality.

Moreover some experimentation suggests that these are (up to conjugation) the only cases where this phenomenon occurs.

3. CONCLUSIONS AND SUMMARY

We have highlighted the importance of specifying the spin structure of the spacetime as part of the data defining a supergravity background by constructing examples of isometric M-theory backgrounds admitting more than one spin structure and preserving a different amount of supersymmetry depending on this choice. Our examples are products of AdS_4 with lens spaces S^7/\mathbb{Z}_{4k} . For $k=1$ the two backgrounds are, respectively, $\frac{1}{4}$ - and $\frac{3}{4}$ -BPS, whereas for $k>1$ they are $\frac{1}{4}$ - and $\frac{1}{2}$ -BPS, respectively. We expect this phenomenon to persist for other backgrounds which are products of AdS_4 with a spherical space form. A systematic analysis of such backgrounds is under way and we will be reporting on these results in a more extensive forthcoming paper [10].

ACKNOWLEDGMENTS

We are grateful to Elmer Rees for useful conversations. JMF would like to thank the IHÉS for hospitality and support during the time it took to complete this work, and in particular Jean-Pierre Bourguignon for the invitation to visit. The research of SG is funded by a PPARC Postgraduate Studentship.

REFERENCES

- [1] P. Freund and M. Rubin, “Dynamics of dimensional reduction,” *Phys. Lett.* **B97** (1980) 233–235.
- [2] W. Nahm, “Supersymmetries and their representations,” *Nucl. Phys.* **B135** (1978) 149–166.
- [3] E. Cremmer, B. Julia, and J. Scherk, “Supergravity in eleven dimensions,” *Phys. Lett.* **76B** (1978) 409–412.
- [4] J. A. Wolf, *Spaces of constant curvature*. Publish or Perish, Boston, Third ed., 1974.
- [5] J. M. Figueroa-O’Farrill and J. Simón, “Supersymmetric Kaluza–Klein reductions of AdS backgrounds,” *Adv. Theor. Math. Phys.* **8** (2004) 217–317. [arXiv:hep-th/0401206](#).
- [6] J. M. Figueroa-O’Farrill, O. Madden, S. F. Ross, and J. Simón, “Quotients of $\text{AdS}_{p+1} \times S^q$: causally well-behaved spaces and black holes,” *Phys. Rev.* **D69** (2004) 124026. [arXiv:hep-th/0402094](#).
- [7] C. Bär, “Real Killing spinors and holonomy,” *Comm. Math. Phys.* **154** (1993) 509–521.
- [8] A. Franc, “Spin structures and Killing spinors on lens spaces,” *J. Geom. Phys.* **4** (1987) 277–287.
- [9] C. Bär, “The Dirac operator on space forms of positive curvature,” *J. Math. Soc. Japan* **48** (1996) 69–83.
- [10] J. M. Figueroa-O’Farrill and S. Gadhia, “Supersymmetric spherical space forms.” in preparation.

SCHOOL OF MATHEMATICS, THE UNIVERSITY OF EDINBURGH, SCOTLAND, UNITED KINGDOM
E-mail address: j.m.figueroa@ed.ac.uk, s.gadhia@sms.ed.ac.uk